L1 Introduction to FEM 1

1.1 Why FEM?

Real life structures are too complex to analyze them analytically. Therefore, certain methods are used to simplify structural analysis. One of these methods is **FEM**.

In the end, one want to predict the behavior of a structure. This can be done by:

1. Testing
2. Modeling

Modeling is generally cheaper and faster.

1.2 Characteristics of FEM

Taking a cantilever beam as an example, one can find the differential equation:

$$\frac{d^2v(x)}{dx^2} = -\frac{M(x)}{EI} \tag{1.1}$$

This DE needs to be solved for equilibrium. This can be done through conservation of energy. The DE is supported by initial and/or boundary conditions, such as:

$$v(x=0) = 0 \tag{1.2}$$

$$\frac{du}{dx}(x=0) = 0 \tag{1.3}$$

An exact solution is possible for simple problems, or with assumptions such as Euler-Bernoulli.

Numerical solutions exist for all problems, with different assumptions, through methods like finite difference or finite element.

Assumption: the behavior is at the nodes. Continuity joins together all nodes in a problem. Disadvantage of concrete methods is the fact that it results in errors.

FEM basic principle:

1. Split structure
2. Assume displacement shape
3. Determine displacement of each node

The second assumption for the shape function is very important.

1.3 FEM Steps and Formulation

In general:

1. Pre-processing
2. Processing
3. Post-processing

1.3.1 Pre-processing

Setting up the problem.

The necessary assumptions are extremely important. Linear models: not large displacements, or material dependencies or nonlinear behavior. Hooke's law is applicable.

① **Discretization**

Dividing the continuum structure

② **Idealization**

Select dimensionality (1D, 2D, 3D), and select the element behavior (axial translation, transverse translation, rotation)

③ **Equilibrium of Each Element**

$$F = Ku \tag{1.4}$$

④ **Assemble the Whole System**

Determine the total stiffness matrix, making use of the connectivity matrix

⑤ **Apply**

Application of loads and boundary conditions

1.3.2 Processing

Solving the problem.

① **Solve the Equilibrium System**

$$F = K^G u \tag{1.5}$$

1.3.3 Post-processing

Some problems require derived results. As an example, the displacement can be used to get the strain or stress.

Assumptions made during pre-processing need to be checked here.

1.3.4 Formulation

Three methods:

1. Direct formulation
2. Minimum total potential energy
3. Weighted residual method

All methods have differences, resulting in different errors.

1.4 Direct Formulation

The example problem is:

1. Discretized into 4 elements and 5 nodes (1D)
2. Idealized to only take axial displacement, and looking at a single element. One can find $F = \frac{EA}{L}\Delta L = k_{eq}\Delta L$.

(3) Set up with the equilibrium equation

Next, the global stiffness matrix can be found. For this:

Each node has a certain displacement which can be found using the previous equations. These will be in the form of an equilibrium equation, resulting in a system of n displacements for n nodes. Combining these into a matrix is the global stiffness matrix, $K^G u = F$.

Instead of applying the reaction force R, one can also apply the fact that this node will have a zero displacement.

Lastly applied loads and boundary conditions are applied.

Processing is done by simply solving the system obtained. Ensure to compare the solution to the exact analytical solution, also keeping in mind on which side it is on (higher or lower).

1.5 Stiffness Matrix by Inspection

The previous method for forming K^G works in general. A quicker method is by adding individual matrices for the individual nodes. Think back to SVV structures, we did this there.

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For a simple spring:

$$F = k\Delta l \tag{2.1}$$
$$F = kx' \tag{2.2}$$
$$\Lambda = \int_0^{x'} F dx' \tag{2.3}$$
$$\Lambda = \int_0^{x'} kx' dx' \tag{2.4}$$
$$\Lambda = \frac{1}{2}kx'^2 = \left(\frac{1}{2}kx'\right)x' \tag{2.5}$$
$$d\Lambda = \frac{1}{2}kx'dx' \tag{2.6}$$
$$kx' = F = \sigma A = \sigma dydz \tag{2.7}$$
$$d\Lambda = \frac{1}{2}\sigma dydzdx' \tag{2.8}$$
$$\varepsilon = \frac{dx'}{dx} \tag{2.9}$$
$$dx' = \varepsilon dx \tag{2.10}$$
$$d\Lambda = \frac{1}{2}\sigma dydz\varepsilon dx \tag{2.11}$$
$$d\Lambda = \frac{1}{2}\sigma\varepsilon dV \tag{2.12}$$
$$d\Lambda = \frac{1}{2}E\varepsilon^2 dV \tag{2.13}$$
$$\Lambda^{(e)} = \int d\Lambda = \int_V \frac{E\varepsilon^2}{2}dV \tag{2.14}$$
$$\Pi = \Lambda - W \tag{2.15}$$
$$\Pi = \sum_{e=1}^{m}\Lambda^{(e)} - \sum_{i=1}^{n}F_i u_i \tag{2.16}$$
$$\tag{2.17}$$

Taking the derivative with respect to u_i leads to n equations and n unknowns, a system which can be solved.

2.1 Minimum Total Potential Energy

The above states the principle for MTPE. With the derivatives taken for Λ, one can also take derivatives for the external work W. Then, taking the total potential energy derivatives with respect to u_i, results in n equations, which can be written as a matrix equation once again.

2.2 Weighted Residual

With a given differential equation for the e.g. displacement, one can put the approximate (e.g. u_{approx}) into the DE, and set the solution to the residual R. The approximate solution has to:

1. Satisfy initial conditions
2. Satisfy boundary conditions
3. Be continuous

With an approximate solution, solve the DE, and find the residual. The Galerkin method uses weighing functions w_i which are orthogonal to the approximate solution. Then:

$$\int_a^b W_i R dx = 0 \tag{2.18}$$

for $i = 1, 2, \ldots$

Note that the number of weighing functions is equal to the number of unknowns. This eventually leads to a system of n equations and n unknowns. Solving the system gives expressions for the c_i used in the approximate solution.

This method gives the values for the constants to minimize the residual, but not for the one closest to the analytical solution.

The weighing functions need to be equal to the shape functions for FEM.

2.3 Shape Functions

In FEM, we want to give a way that the system behaves by enforcing deflection methods. These methods are shape functions.

Starting with assumed displacement, we have 2 nodes per element, and one DOF per node. We can then fill in the known displacements, and we can solve for the constants in the equations. Then rewriting in terms of u, gives us the shape functions. The number of shape functions for all nodes is equal to the degrees of freedom (2 nodes, 1DOF equals 2 shape functions).

Shape functions are always 1 at their own node, and 0 at the other. The sum of the shape functions at any point is 1.

2.4 Connectivity Matrix

We already know that we can assemble the global stiffness matrix by adding the element stiffness matrices. The connectivity matrix shows where to add the individual element stiffness matrices. Row n gives start and end node of element n. This automates the process of going from element to global stiffness matrix.

L3 Truss Elements

Members in truss structures do not have bending, only axial loads. When making a cut, we get internal forces which we can determine. If bending can not be neglected, we have to use beam elements, either 1D or 2D.

Multiple types of elements exist.

1. 0D elements are points, point masses, springs, ...
2. 1D elements are wires, and truss or 1D beam elements
3. 2D elements are planes ($t << w, h$)
4. 3D elements are 3D objects

Combinations of element types within a structure are possible.

Any structural element can be represented by a truss element. In FEM, a statically determinate and indeterminate structure can be solved in the same way. As derived before:

$$k_{\text{eq}} = \frac{EA}{l} \tag{3.1}$$

3.1 Truss Element Transformations

A transformation indicates a change from one coordinate system to another. The global coordinate system is equal for each element. Then each element also has their own local coordinate system. For a general element at an angle θ in the $(x - y)$ plane:

$$u_{jx} = u_{jx'} \cos\theta - u_{jy'} \sin\theta \tag{3.2}$$
$$u_{jy} = u_{jx'} \sin\theta + u_{jy'} \cos\theta \tag{3.3}$$
$$\tag{3.4}$$

We can put this in matrix formulation:

$u_G = (u_{ix}, u_{iy}, u_{jx}, u_{jy})$, u_L local.

$u_G = [T]u_L$, we can easily find $[T]$

Forces also need to be rotated. Same transformation matrix.

Always check extreme cases (0 and 90 degrees).

3.2 Transformation of Equilibrium

When extending the number of degrees of freedom, only add 0's in the stiffness matrix (see video for explanation and example).

We have a $F_L = [K]_L u_L$

$$[T]^{-1} F_G = [K]_L [T]^{-1} u_G \tag{3.5}$$
$$[T][T]^{-1} F_G = [T][K]_L [T]^{-1} u_G \tag{3.6}$$
$$F_G = [T][K]_L [T]^{-1} u_G \tag{3.7}$$
$$[K]_G = [T][K]_L [T]^{-1} \tag{3.8}$$

L4 Beam and Frame Elements

4.1 Beam Theory

Truss:

1. No bending
2. Only in-plane deformation
3. No out-of-plane deformation

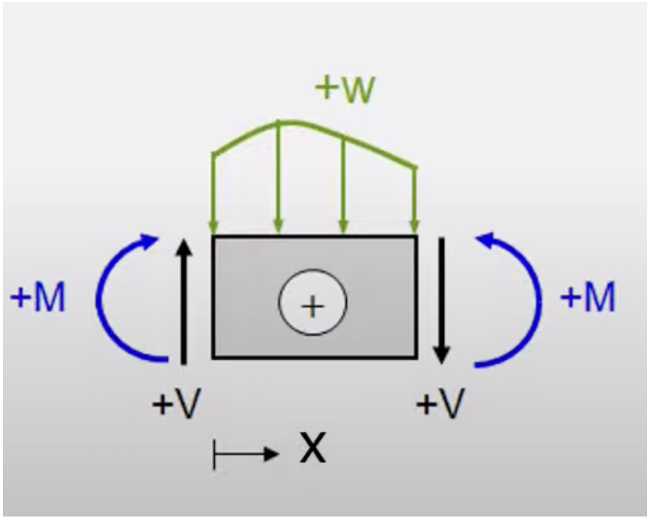
Beam:

1. Only bending
2. No in-plane deformation
3. Only out-of-plane deformation

The nodes have a deformation angle θ, a moment M, and a shear force V.

4.1.1 Euler-Bernoulli

Coordinate system (y down):



Then:

$$EI\frac{d^2v}{dx^2} = -M(x) \tag{4.1}$$

$$\sigma = \frac{My}{I} \tag{4.2}$$

$$EI\frac{d^3v}{dx^3} = \frac{dM(x)}{dx} = -V(x) \tag{4.3}$$

$$EI\frac{d^4v}{dx^4} = \frac{-dV(x)}{dx} = w(x) \tag{4.4}$$

We will use these for deriving equilibrium.

4.2 Beam Elements

4.2.1 Shape Functions

We first need to assume a displacement. Since we have four DoF, we need four constants in the displacement function.

$$v = c_1 + c_2x + c_3x^2 + c_4x^3 \tag{4.5}$$

We can say that node i lies at the origin, and the beam has a length l. Then:

$$x_i = 0 \tag{4.6}$$
$$x_i = l \tag{4.7}$$

Then:

$$u_{i1} = v(x=0) = c_1 \tag{4.8}$$
$$u_{i2} = \frac{dv}{dx}(x=0) = c_2 \tag{4.9}$$
$$u_{j1} = v(x=l) = c_1 + c_2l + c_3l^2 + c_4l^3 \tag{4.10}$$
$$u_{j2} = \frac{dv}{dx}(x=l) = c_2 + 2c_3l + 3c_4l^2 \tag{4.11}$$

We have four equations and four unknowns, which can be solved. We get:

$$c_1 = u_{i1} \tag{4.12}$$
$$c_2 = u_{i2} \tag{4.13}$$
$$c_3 = -3\frac{u_{i1}}{l^2} - 2\frac{u_{j1}}{l^2} + 3\frac{u_{j1}}{l^2} - \frac{u_{j2}}{l} \tag{4.14}$$
$$c_4 = 2\frac{u_{i1}}{l^3} + \frac{u_{i2}}{l^2} - 2\frac{u_{j1}}{l^3} + \frac{u_{j2}}{l^2} \tag{4.15}$$

For the shape function, we need the displacement function as a function of the degrees of freedom:

$$v = N_1u_{i1} + N_2u_{i2} + N_3u_{j1} + N_4u_{j2} \tag{4.16}$$
$$N_1 = \frac{l^3 - 3lx^2 + 2x^3}{l^3} \tag{4.17}$$
$$N_2 = \frac{l^2x - 2x^2l + x^3}{l^2} \tag{4.18}$$
$$N_3 = \frac{3x^2l - 2x^3}{l^3} \tag{4.19}$$
$$N_4 = \frac{x^3 - x^2l}{l^2} \tag{4.20}$$

4.2.2 Stiffness Matrix

The strain of an element is given as:

$$\epsilon = \frac{du}{dx} = y\frac{d^2v}{dx^2} \tag{4.21}$$

Then:

$$\Lambda^{(e)} = \int_V \frac{E\epsilon^2}{2}dV \tag{4.22}$$
$$\Lambda^{(e)} = \frac{E}{2}\int_V (y\frac{d^2v}{dx^2})^2dV = \frac{E}{2}\int_0^l (\frac{d^2v}{dx^2})^2dx\int_A y^2dA \tag{4.23}$$
$$\Lambda^{(e)} = \frac{EI}{2}\int_0^l (\frac{d^2v}{dx^2})^2dx \tag{4.24}$$
$$\frac{d^2v}{dx^2} = \frac{d^2}{dx^2}\begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}\begin{bmatrix} u_{i1} \\ u_{i2} \\ u_{j1} \\ u_{j2} \end{bmatrix} \tag{4.25}$$

Then, define:

$$[B] = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \tag{4.26}$$
$$B_1 = (\frac{-6}{l^2} + \frac{12x}{l^3}) \tag{4.27}$$
$$B_2 = (\frac{-4}{l} + \frac{6x}{l^2}) \tag{4.28}$$
$$B_3 = (\frac{6}{l^2} - \frac{12x}{l^3}) \tag{4.29}$$
$$B_4 = (\frac{6x}{l^2} - \frac{2}{l}) \tag{4.30}$$

Such that:

$$\frac{d^2v}{dx^2} = [B][u] \tag{4.31}$$
$$\tag{4.32}$$

Then:

$$(\frac{d^2v}{dx^2})^2 = [u]^T[B]^T[B][u] \tag{4.33}$$
$$\frac{\partial \Lambda^{(e)}}{\partial u} = EI\int_0^l [B]^T[B]dxu \tag{4.34}$$

The final solution is:

$$\frac{\partial \Lambda^{(e)}}{\partial u} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_{i1} \\ u_{i2} \\ u_{j1} \\ u_{j2} \end{bmatrix} \tag{4.35}$$
$$[K]^{(e)} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{4.36}$$

4.2.3 Equilibrium

Assuming a constant distributed load w along the beam. Then:

$$W_p = \int_0^l wvdx \tag{4.37}$$
$$W_p = w\int_l \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}[u]dx \tag{4.38}$$
$$W_p = w\int_l [N]dx[u] \tag{4.39}$$
$$W_p = w\begin{bmatrix} \frac{l}{2} & \frac{l^2}{12} & \frac{l}{2} & -\frac{l^2}{12} \end{bmatrix}[u] \tag{4.40}$$

To get the force:

$$F_{i1} = F_i = -\frac{\partial W_p}{\partial u_{i1}} = -\frac{wl}{2} \tag{4.41}$$

$$F_{i2} = M_i = -\frac{\partial W_p}{\partial u_{i2}} = -\frac{wl^2}{12} \tag{4.42}$$

$$F_{j1} = F_j = -\frac{\partial W_p}{\partial u_{j1}} = -\frac{wl}{2} \tag{4.43}$$

$$F_{j2} = M_j = -\frac{\partial W_p}{\partial u_{j2}} = \frac{wl^2}{12} \tag{4.44}$$

$$\tag{4.45}$$

And thus:

$$[F] = \begin{bmatrix} -\frac{wl}{2} \\ -\frac{wl^2}{12} \\ -\frac{wl}{2} \\ \frac{wl^2}{12} \end{bmatrix} \tag{4.46}$$

From that, the equilibrium condition is:

$$[K]^{(e)}[u] = [F] \tag{4.47}$$

$$\frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \\ u_{j1} \\ u_{j2} \end{bmatrix} = \begin{bmatrix} -\frac{wl}{2} \\ -\frac{wl^2}{12} \\ -\frac{wl}{2} \\ \frac{wl^2}{12} \end{bmatrix} \tag{4.48}$$

4.3 Frame Elements

Frame elements are elements which can both bend and axially deform.

1. Bending
2. In-plane deformation
3. Transverse deformation

Note: in most FEM software, a Beam matches this definition.

Each node has 3 DoF, and we have forces in x and y directions, and a moment.

Since we have 3×2 DoF in one element, we might need 6 constants in the shape function. However, the deformations are decoupled. As a result, we can superpose the two elements.

4.3.1 Stiffness Matrix

For the beam:

$$[K]_{beam} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{4.49}$$

We need to extend this to a 6×6 matrix. We have:

$$[u] = \begin{bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{bmatrix} \tag{4.50}$$

Then:

$$[K]_{beam} = \frac{EI}{l^3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} \tag{4.51}$$

For a truss:

$$[K]_{truss} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.52}$$

Then:

$$[K]_{truss} = \frac{EA}{l} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.53}$$

With the two 2×2 matrices defined, we can superpose the two to get the final stiffness matrix:

$$[K]_{frame} = \frac{EA}{l}[K]_{truss} + \frac{EI}{l^3}[K]_{beam} \tag{4.54}$$

4.3.2 Transformation Matrix

We have for a truss:

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{4.55}$$

Extending this to also add the θ displacements in $[u]$:

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.56}$$

L5 Higher Order Elements

5.1 Higher Order Elements

Assuming a certain displacement function, a linear element might not match all too well. We can solve this by either adding more elements, or moving to higher order elements.

5.1.1 Quadratic Elements

The assumed displacement has to be of the form:

$$u^{(e)} = c_1 + c_2 x + c_3 x^2 \tag{5.1}$$

As we only have 2 DoF, we need another internal node in the element to find all constants, giving us an extra DoF. We match the displacement at this node. The extra node does not mean splitting the element in two: we still have one element. We have three known displacements: u_i, u_j, u_k. We can then assume the origin is at node i, and the element has length l. Then, node j is at $l/2$ and node k is at l. Substituting and solving for c_i gives:

$$\vec{c} = \begin{bmatrix} u_i \\ \frac{1}{l}(-3u_i + 4u_j - u_k) \\ \frac{4}{l^2}(\frac{u_i}{2} - u_j + \frac{u_k}{2}) \end{bmatrix} \tag{5.2}$$

We now want to find the shape functions:

$$u^{(e)} = N_i u_i + N_j u_j + N_k u_k \tag{5.3}$$

Solving:

$$[N] = \begin{bmatrix} \frac{2}{l^2}(x - x_j)(x - x_k) \\ \frac{-4}{l^2}(x - x_i)(x - x_k) \\ \frac{2}{l^2}(x - x_i)(x - x_j) \end{bmatrix} \tag{5.4}$$

Changing the order of the element changes the way the element behaves.

The internal nodes do not affect the mesh of the structure.

Note that a higher order element is not always a better option than using multiple linear elements. To find out which option is better, one needs to perform a convergence study.

5.2 Lagrange Interpolations Functions

Lagrange interpolation functions are a shortcut to find the shape functions. Before, we start with the polynomial, and found the constants. Then, the shape functions were found using the displacements. In general, we get the equation:

$$N_i = \Pi_{m=1}^{n} \frac{(x - x_m) \text{ omitting } (x - x_i)}{(x_i - x_m) \text{ omitting } (x_i - x_i)} \tag{5.5}$$

5.3 Natural Coordinates

Natural coordinates are the third coordinate system used in FEM. The system is defined by the ξ and η coordinates. The origin coincides with the middle of the element. Node i is at $\xi = -1$, and node j is at $\xi = 1$. We see (local $\leftrightarrow$ natural):

$$\xi = \frac{2x'}{l} - 1 \tag{5.6}$$

$$x' = \frac{l(xi + 1)}{2} \tag{5.7}$$

Then:

$$N_i = 1 - \frac{x'}{l} \tag{5.8}$$

$$N_i = 1 - \frac{(\xi + 1)}{2} \tag{5.9}$$

$$N_i = \frac{1}{2}(1 - \xi) \tag{5.10}$$

And:

$$N_j = \frac{x'}{l} \tag{5.11}$$

$$N_j = \frac{(\xi + 1)}{2} \tag{5.12}$$

$$N_j = \frac{1}{2}(1 + \xi) \tag{5.13}$$

The advantage is that the natural coordinates are independent of the element length. Another advantage is that we can still use Lagrange interpolation.

5.4 Isoparametric Elements

Finding a local coordinate system is not useful in isoparametric elements. We can reshape such an element to a square element in natural coordinates.

Mainly used for higher dimensionality elements.

L6 Triangular Elements

6.1 Triangular Membrane Elements

For a triangular membrane element, we use the standard $x-y$-coordinate system. A membrane element means there is only in-plane deformation. We thus have displacements in x-direction (u or u_x) and in y-direction (v or u_y).

6.1.1 Shape Functions

We need to assume a displacement function for x and y. We use a linear displacement function. We also assume no relation between x and y. Resulting is:

$$u^{(e)} = c_1 + c_2 X + c_3 Y \tag{6.1}$$
$$u^{(e)} = c_4 + c_5 X + c_6 Y \tag{6.2}$$

Filling in the known points i, j, k, results in 6 equations with the 6 c_i and the displacements unknown. The displacements in x and y directions are found very similarly, and thus in the following only x is considered.

Solving for the constants:

$$c_1 = \frac{1}{2A}\left[\left(X_j Y_k - X_k Y_j\right) u_i + \left(X_k Y_i - X_i Y_k\right) u_j + \left(X_i Y_j - X_j Y_i\right) u_k\right] \tag{6.3}$$
$$c_2 = \frac{1}{2A}\left[\left(Y_j - Y_k\right) u_i + \left(Y_k - Y_i\right) u_j + \left(Y_i - Y_j\right) u_k\right] \tag{6.4}$$
$$c_3 = \frac{1}{2A}\left[\left(X_k - X_j\right) u_i + \left(X_i - X_k\right) u_j + \left(X_j - X_i\right) u_k\right] \tag{6.5}$$
$$\text{with } A = \frac{1}{2}\left[X_i\left(Y_j - Y_k\right) + X_j\left(Y_k - Y_i\right) + X_k\left(Y_i - Y_j\right)\right] \tag{6.6}$$

Rewriting to find the shape functions:

$$u^{(e)} = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} \tag{6.7}$$

We can find that:

$$N_i = \frac{1}{2A}\left[\alpha_i + \beta_i X + \delta_i Y\right] \tag{6.8}$$
$$N_j = \frac{1}{2A}\left[\alpha_j + \beta_j X + \delta_j Y\right] \tag{6.9}$$
$$N_k = \frac{1}{2A}\left[\alpha_k + \beta_k X + \delta_k Y\right] \tag{6.10}$$

These are quite identical for y. We thus get:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{bmatrix} U_{ix} \\ U_{iy} \\ U_{jx} \\ U_{jy} \\ U_{kx} \\ U_{ky} \end{bmatrix} \tag{6.11}$$

6.1.2 Stiffness Matrix

We assume:

1. In-plane only
2. Plane stress

We use the strain energy, which defines the stiffness matrix, and thus needs to be minimized. We have:

$$\Lambda^{(e)} = \frac{1}{2}\int_V (\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \tau_{xy}\gamma_{xy})\, dV \tag{6.12}$$

We can write the strain vector as:

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \tag{6.13}$$

The stress vector:

$$[\sigma] = [D][\epsilon] \tag{6.14}$$

Here, $[D]$ is for an isotropic material:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{6.15}$$

We can then find:

$$\Lambda^{(e)} = \frac{1}{2}\int_V [\epsilon]^T [D][\epsilon] dV \tag{6.16}$$

We need to find an expression that relates the strain and the displacement as a matrix. We get:

$$[\epsilon] = [B][U] \tag{6.17}$$

with:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_k & 0 \\ 0 & \delta_i & 0 & \delta_j & 0 & \delta_k \\ \delta_i & \beta_i & \delta_j & \beta_j & \delta_k & \beta_k \end{bmatrix} \tag{6.18}$$

We can then get:

$$\Lambda^{(e)} = \frac{1}{2}\int_V [U]^T [B]^T [D][B][U] dV \tag{6.19}$$

And then:

$$[K]^{(e)} = V[B]^T[D][B] \tag{6.20}$$

6.1.3 Force Vector

Using the principle of minimum total potential energy, we have to take the partial derivatives of the work done to obtain the force vector. Considering point loads:

$$W^{(e)} = \int_A [U]^T [P] dA \tag{6.21}$$

Then:

$$[F]^{(e)} = \begin{bmatrix} F_{ix} \\ \vdots \end{bmatrix} \tag{6.22}$$

For a distributed load:

$$W^{(e)} = \int_A [N]^T [U]^T [P] dA \tag{6.23}$$

Then:

$$[F]^{(e)}. = \int_A [N]^T [P] dA = \frac{tl_{ij}}{2}\begin{bmatrix} p_x \\ p_y \\ p_x \\ p_y \\ 0 \\ 0 \end{bmatrix} \tag{6.24}$$

We can then form the equilibrium equation quite easily.

6.2 Triangular Bending Element

Assumptions:

1. Thickness is small compared to the other dimensions
2. Deflections are small
3. Mid-plane does not go deformation (pure bending)
4. No transverse shear deformation

We have:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \tag{6.25}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \tag{6.26}$$

$$\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{6.27}$$

6.2.1 Shape Functions

We have out of plane displacements for each of the three nodes, and the bending in x and y directions. We thus get 9 DoF. We assume the following displacement functions (based on Pascal's triangle):

$$w(x,y) = c_1 + c_2 x + c_3 y + c_4 xy + c_5 x^2 + c_6 y^2 + c_7\left(x^2 y + xy^2\right) + c_8 x^3 + c_9 y^3 \tag{6.28}$$

Assuming the origin is at i:

$$w_i = c_i \tag{6.29}$$

$$\frac{\partial w_i}{\partial x} = c_2 \tag{6.30}$$

$$\frac{\partial w_i}{\partial y} = c_3 \tag{6.31}$$

Doing this for the other nodes as well gives all constants. Next, we need to find the shape functions by rewriting. Then, we use the strain energy to find the stiffness matrix, after which we can formulate the strain-displacement matrix $[B]$ as:

$$[B] = -t\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix} \tag{6.32}$$

$[D]$ is given by the same equation. We can then get the stiffness matrix.

L7 Rectangular Elements

We still use the thin plate theory as for the previous lecture.

7.1 Shape Functions in Cartesian Coordinates

We have to start with the displacement function. We have 2 displacements at each of the four nodes. Again, we find that the shape functions are identical for u and v. For u:

$$u^{(e)} = c_1 + c_2 X + c_3 Y + c_4 XY \tag{7.1}$$

Filling in the known displacements at the nodes, we can find the constants as:

$$c_1 = u_i \tag{7.2}$$
$$c_2 = \frac{1}{l}\left(u_j - u_i\right) \tag{7.3}$$
$$c_3 = \frac{1}{w}\left(u_n - u_i\right) \tag{7.4}$$
$$c_4 = \frac{1}{lw}\left(u_i - u_j + u_m - u_n\right) \tag{7.5}$$

Then, the shape functions can be found as:

$$N_i = \left(1 - \frac{x}{l}\right)\left(1 - \frac{y}{w}\right) \tag{7.6}$$
$$N_j = \frac{x}{l}\left(1 - \frac{y}{w}\right) \tag{7.7}$$
$$N_m = \frac{x}{l}\frac{y}{w} \tag{7.8}$$
$$N_n = \frac{y}{w}\left(1 - \frac{x}{l}\right) \tag{7.9}$$

7.2 Shape Functions in Natural Coordinates

We switch to natural coordinates as elements are not always perfect rectangles. See Lecture 5 for the definition.

For the following, we assume a rectangle, but methodology is the same for irregular shapes. We have:

$$\xi = \frac{2x}{l} - 1 \tag{7.10}$$
$$\eta = \frac{2y}{w} - 1 \tag{7.11}$$
$$x = \frac{l}{2}(\xi + 1) \tag{7.12}$$
$$y = \frac{w}{2}(\eta + 1) \tag{7.13}$$

Then, plugging them in into the shape functions:

$$N_i = \frac{1}{4}(1 - \xi)(1 - \eta) \tag{7.14}$$
$$N_j = \frac{1}{4}(1 + \xi)(1 - \eta) \tag{7.15}$$
$$N_m = \frac{1}{4}(1 + \xi)(1 + \eta) \tag{7.16}$$
$$N_n = \frac{1}{4}(1 - \xi)(1 + \eta) \tag{7.17}$$

7.3 Isoparametric Elements

In natural coordinates, isoparametric elements are rectangles with lengths of 2. We have:

$$u = \sum_i N_i U_{ix} \quad (7.18)$$

$$v = \sum_i N_i U_{iy} \quad (7.19)$$

7.4 Derivatives in Natural Coordinates

Just chain rule honestly:

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \quad (7.20)$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} \quad (7.21)$$

$$(7.22)$$

We can put that into matrix form, and we get the Jacobian:

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \quad (7.23)$$

We use it like:

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = [J]^{-1} \begin{bmatrix} f_\xi \\ f_\eta \end{bmatrix} \quad (7.24)$$

7.5 Stiffness Matrix

We start with the strain energy. From the triangular element derivation:

$$\Lambda^{(e)} = \frac{1}{2}\int_V [\epsilon]^T [D][\epsilon] dV = \frac{1}{2} t_e \int_A [\epsilon]^T [D][\epsilon] dA \quad (7.25)$$

We have again three parts of the strain vector, equal to the equations in triangular elements. We get:

$$\epsilon_{xx} = \frac{1}{\det J}\left(J_{22}\frac{\partial u}{\partial \xi} - J_{12}\frac{\partial u}{\partial \eta}\right) \quad (7.26)$$

and so forth. We can get the matrix expression:

$$[\epsilon] = \frac{1}{\det J}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \quad (7.27)$$

The partial derivatives can easily be found from the shape functions. Combining everything:

$$[\epsilon] = [A][M][U] = [B][U] \quad (7.28)$$

Then:

$$\Lambda^{(e)} = \frac{1}{2} t_e \int_A [U]^T [B]^T [D][B][U] dA \quad (7.29)$$

Using the natural coordinates for the integral results in:

$$[K]^{(e)} = t_e \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B] \det J d\xi d\eta \quad (7.30)$$

7.6 Gauss Quadrature

A method to numerically perform an integral. Honestly doesn't really matter, just some background info as this is what FEM uses in the background as well.